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On Analytic Solutions of Elastic Net Displacements around a Circular Tunnel

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Abstract

Displacements around a tunnel, occurring as a result of excavation, consist of the elastic and plastic parts. In this paper, we discuss the elastic part of displacements as a result of excavation, called net displacement. In general, the previous analytical solutions presented for determining the displacements around a circular tunnel in an elastic medium do not give the net displacements directly. The well-known Kirsch solution is the most widely used method for determining the induced stresses and net displacements around a circular opening in a biaxially-loaded plate of homogeneous, isotropic, continuous, linearly elastic material. However, the complete solution for obtaining the net displacements has not been presented or highlighted in the available literature. Using the linear elasticity, this paper reviews and presents three different analytical methods for determining the net displacements directly as well as induced stresses around a circular tunnel. The three solution methods are the Lame' method, airy stress function method, and complex variable method. The tunnel is assumed to be situated in an elastic, continuum, and isotropic medium in the plane strain condition. The solutions are presented for both the hydrostatic and non-hydrostatic in situ stresses in the 2D biaxial loading condition along with an internal pressure. Loading and unloading in tunneling occurring as a result of excavation and stress differences between the induced and initial ones are considered to evaluate the net displacements directly. Finally, some examples are given to demonstrate the complete solution and show the difference between the net elastic displacements as a result of excavation and total elastic displacements that are not real.

1. Introduction

Determining the induced stresses and displacement fields around a circular opening (for example a circular tunnel) as a result of excavation has always been an interesting topic in tunneling, mining, petroleum, and civil engineering. Although the numerical methods such as the finite element method are often used these days to calculate the stresses and displacements around openings, the analytical methods (mainly by use of linear elasticity) for simplified shapes such as cylindrical (circular) excavation are highly useful in understanding the effect of a particular parameter on the results (for instance, the effect of rock mass deformation modulus on displacements around a circular opening). Although in reality, rock masses are not actually linearly elastic, a rock mass behaves as linearly elastic for incremental stress changes. This approximation in rock mechanics is accurate enough for solving the problems like determining the stress-strain relationships in a rock mass.

In 1898, Kirsch has presented equations for determining the stresses in an infinite plate with a circular hole located in a biaxially-loaded homogeneous, isotropic, continuous, and linearly elastic material. This analytical solution has been widely used in different geotechnical and rock mechanical problems, and it is also much treated in the textbooks of rock mechanics and elasticity theory [1, 2]. Using the equations, a deep understanding of the physics related to the rock mechanical issues can

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be achieved. However, in most textbooks [3–7], only the last equations are presented and the procedure for obtaining the equations is not given comprehensively in detail.

Although the procedure for obtaining the stress and displacement fields around a circular hole (hydrostatic or non-hydrostatic in situ stress conditions) have been presented in some other reference books [2, 8–13], the net displacements around a circular tunnel (the displacements as a result of excavation only) has not been highlighted or even considered in these books. In fact, the equations presented for evaluating the displacements around a circular tunnel differ from the Kirsch equations. On the other hand, the total displacements that consider the in situ stress effect before excavation is taken in to account, which can be misleading in tunneling.

Determination of ground displacements produced by excavation of circular opening (net displacements as a result of excavation only) is one of the most important problems associated with tunneling. As a case in example, an effective method for monitoring an underground opening is to measure the relative displacements of points on the walls at surface or at different depths. Considering the fact that elastic displacement is some part of total displacement and to interpret such data, it is helpful and necessary to know the magnitudes of displacements associated with the elastic behavior [5,14].

The problem of a circular opening in an infinite continuum medium and the stress and displacements fields around the opening as a result of excavation (tunneling) includes both the loading and unloading in the tangential and radial directions, respectively. Actually, it is the stress differences (between the original one and the current induced one that may be increased or decreased) that cause the net displacements, not the current induced stress field. This is a very subtle key point in determining the net displacements.

The reference textbooks such as those by Timoshenko and Goodier [15] may be used to fully understand the elasticity and related methods of the analysis of stress and strain around a circular opening. However, it should be noted that after excavation, only net displacements must be considered in the analysis related to tunneling. For example, in the analysis of the rock bolt-ground interaction, the net displacements will affect the rock bolts since they are installed after excavation, and the initial displacements originated by the constant far field stress do not affect the rock bolts. This has been considered by some researchers such as Bobet [16]. Also in evaluating the stresses and displacements around a circular tunnel in the elastic-plastic conditions, the elastic net displacements that are some part of the total displacements should be used in the analysis. In many published papers [14, 16–30], the net displacements have been used in the analyses, mainly in the hydrostatic in situ stress condition. However, the procedure for obtaining the net displacements has not been explained in detail.

Einstein and Schwartz [31] have presented a solution for the net displacements of tunnel as a result of excavation only. They called this displacement the incremental displacement, which has been evaluated by subtracting the displacement as a result of initial stresses from the one as a result of the current induced stresses. Bobet [16] has also used the same procedure in the elastic solution for deep tunnels. This procedure may not be convenient in the stress-strain analysis. Thus a direct solution for obtaining the net displacement is presented in this paper.

In this work, three methods were used for determining the net displacements induced as well as the induced stresses around a circular opening in the elastic condition. These methods are as follow:

A. Lame' method (hollow cylinder using strength of material);
B. Airy stress function method;
C. Complex variable method.

Although most of the mathematical skills and solutions may be found in the reference textbooks involving elasticity, these methods are reviewed and presented in detail to evaluate directly the net displacements around a circular tunnel by emphasis on the loading and unloading around circular opening as a result of excavation. Actually it is emphasized that in tunneling instead of only the mechanical viewpoint, the ground real condition due to the existence of in situ stresses before excavation should be considered (rock mechanical instead of only mechanical viewpoint).

All the three methods are used for the hydrostatic in situ stress condition, and the last two methods are used for the non-hydrostatic in situ stress condition. Finally, calculating the net displacements is discussed in each method and some examples are presented for clarity.

2. Problem definition and assumptions

A circular tunnel with internal radius, a, located in an elastic, homogenous, isotropic material in the plane strain condition is considered. Both the hydrostatic in situ stress (Figure 1) and non-hydrostatic in situ stress (Figure 2) conditions are taken in to account in the analyses.

In the hydrostatic in situ stress condition, before excavation, the in situ (initial) stress P_o exists. After excavation, the amount of P_o will decrease and increase in the radial and tangentially directions, respectively, in the vicinity of the tunnel (induced stresses will emerge). The inner pressure P_i may exist due to the tunnel face confinement effect (along the tunnel axis), support pressure or internal fluid pressure. The plane strain condition is assumed in this problem, which means that the strains occur in the plane shown in Figure 1. Moreover, the problem is axi-symmetric, meaning that the radial and tangential induced stresses (σ_r, σ_θ) in the rock mass around tunnel are principal stresses. Assuming that the stresses along the tunnel axis remain the mean principal stresses, σ_r and σ_θ will be the minimum and the maximum principal stresses, respectively. In the non-hydrostatic in situ stress condition, before excavation, the in situ stresses in the horizontal and vertical directions are σ_h and σ_v , which differ from each other in two directions. The parameter K is defined as the horizontal to vertical stress ratio (initially).

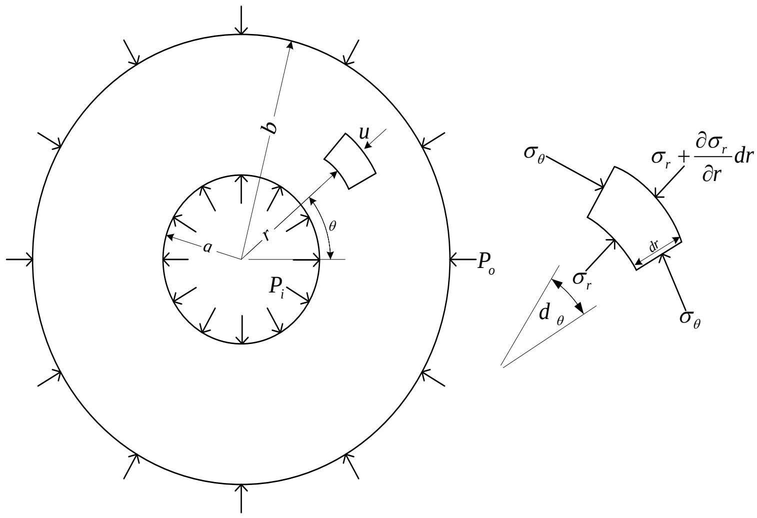

Figure 1. Problem geometry in hydrostatic in situ stress condition.

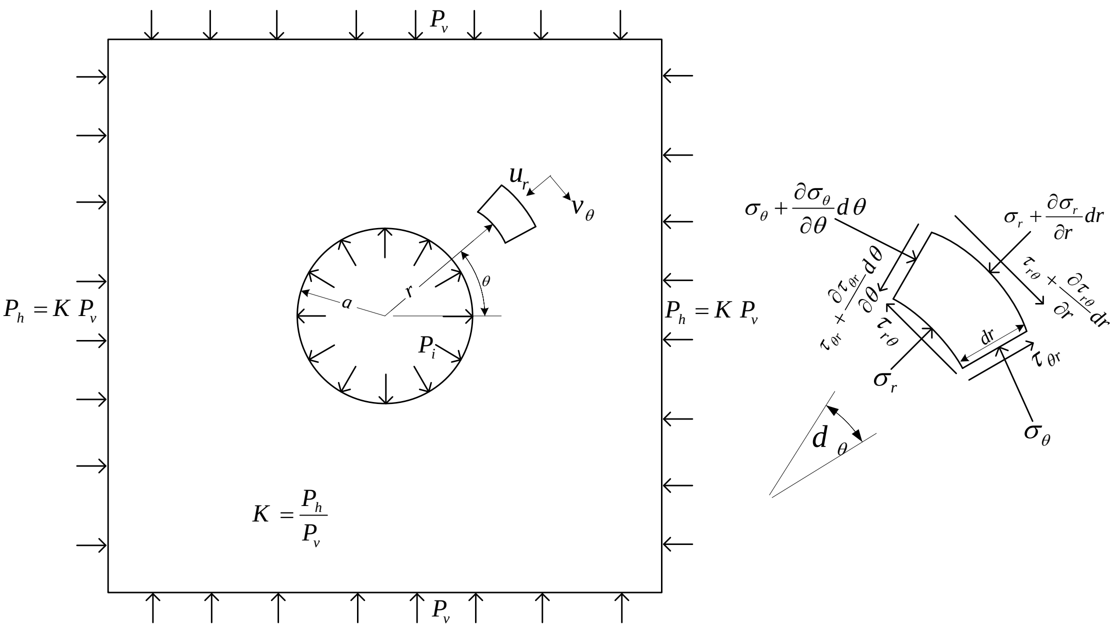

Figure 2. Problem geometry in non-hydrostatic in situ stress condition.

The main purpose in this problem is to obtain directly the elastic net displacements around a circular tunnel using the elasticity and different analytical methods in the hydrostatic and non-hydrostatic stress conditions. In this problem, the compression pressures (stresses) are assumed to be positive and the displacements inward the excavation are considered positive. This is an assumption in this problem and, in some other references, the positive directions of displacements are assumed in the opposite directions (thus the final equation differs in a negative sign).

3. Solutions for hydrostatic in situ stress

In the hydrostatic in situ stress condition, three methods will be used to solve the problem, as follow:
A- Lame' solution (hollow cylinder using strength of material):
In this method, using the stress equilibrium equations, strain compatibility, and properties of material, a differential equation is obtained. Then using the boundary conditions, the differential equation is solved and the radial and tangential stresses around tunnel are evaluated. Then considering the strain-stress relations in the plane strain condition and axi-symmetric assumption, the net displacements around the tunnel are determined as a result of excavation. This problem was originally solved by Lame in 1833 and referred to as the Lame' problem [10, 11]. In this work, the stresses and net displacements will be evaluated in an infinite medium (infinite outer boundary, $b \to \infty$).
B- Airy stress function method:
In this method, a stress function (called the Airy stress function) that satisfies the biharmonic compatibility equation is selected. Based on this function, the stresses around an opening (tunnel) are determined. Then using the stress-strain relations and the boundary condition, the net displacements are defined.
C- Complex variable method:
This method provides a very powerful tool for the solution of many problems in elasticity. In this method, the stresses and displacements are defined based on two complex variable functions that satisfy the equilibrium equations and the Hook's law. It should be mentioned that due to the geometry of the problem, the cylindrical (polar) coordinate will be used to solve the problem. Since the cylinder is long, every ring of unit thickness measured perpendicular to the plane of paper is stress alike [10]. For the case of tunneling in infinite media, the limit of radius b will be considered as infinite. Moreover, the gravity is not considered to maintain the axi-symmetric condition.

3.1 Lame' solution (hollow cylinder using strength of material)

Although the Lame's solution is presented in the reference books published by Popove [10] and Ameen [11], a detailed solution is reviewed and presented here with the geological, rock mechanics, and tunnelling viewpoint to calculate the net displacements as a result of excavation. Each infinitesimal element selected in a ring around a circular opening should be in a static equilibrium. By summing the forces and moments on a 2D element (as shown in Figure 1), the stress equilibrium equation can be obtained as:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

where σ_r and σ_θ are the radial and tangential stresses, respectively, and r is the distance of the element from the tunnel center. This equation has two unknown stresses and requires to be written based on one unknown, so it can be solved.
The radial and tangential strains occur as a result of radial displacement. Due to the axi-symmetric nature of the problem, the strain-displacement relations can be written as:

$$\varepsilon_\theta = \frac{u}{r},\ \varepsilon_r = \frac{du}{dr} \tag{2}$$

where ε_r and ε_θ are the radial and tangential strains, respectively, and u is the radial displacement in the distance r from the tunnel center. The generalized Hooke's law relating strains to stress is given in the following expressions:

$$\begin{aligned}
\varepsilon_r &= \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] \\
\varepsilon_\theta &= \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] \\
\varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_r + \sigma_\theta)]
\end{aligned} \tag{3}$$

where σ_z is the stress along the tunnel axis, and E and ν are the elastic modulus and Poisson's ratio of rock material. Due to the plane strain condition, $\varepsilon_z = 0$, which means $\sigma_z = \nu(\sigma_r + \sigma_\theta)$. Thus by recalculating and reordering the equations, it can be written that:

$$\begin{aligned}
\sigma_r &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_r + \nu\varepsilon_\theta] \\
\sigma_\theta &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_r + (1-\nu)\varepsilon_\theta]
\end{aligned} \tag{4}$$

Substitution of Equation (2) in Equation (4) and then replacing the result in Equation (1), the following differential equation based on the radial displacement variable can be achieved:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{5}$$

This is a second-order homogenous ordinary differential equation known as the Euler–Cauchy Equation, which can be converted into one with constant coefficient by substituting. Then the answer of the resulting constant coefficient differential equation can be obtained by solving an algebraic equation known as the characteristic equation [32, 33]. Finally, the general solution of Equation (5) will be in the following form:

$$u = C_1\, r + \frac{C_2}{r} \tag{6}$$

where C_1 and C_2 are constants, which should be determined from the boundary conditions. The point is that the displacement of the excavation boundary is unknown and should be determined (unknown boundary condition). On the other hand, the external and internal pressures (stresses) are known. Thus the known boundary conditions are: $r = a \Rightarrow \sigma_r = P_i$ & $r = b = \infty \Rightarrow \sigma_r = P_o$.

Based on Equation (6), it can be written that: $\frac{du}{dr} = C1 - \frac{C2}{r^2}$ Then by considering Equations (2) and (4) and the known boundary conditions, the following relations can be written:

$$\sigma_{r(r=a)} = P_i = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\left(C_1 - \frac{C_2}{a^2}\right) + \frac{\nu}{a}\left(C_1\, a + \frac{C_2}{a}\right)\right] \tag{7}$$

$$\sigma_{r(r=b\to\infty)} = P_o = \frac{E}{(1+\nu)(1-2\nu)}C_1 \tag{8}$$

Solving the equations simultaneously for C_1 and C_2 yields:

$$C_2 = \frac{(1+\nu)}{E}(P_o - P_i)a^2$$
$$C_1 = \frac{(1+\nu)(1-2\nu)}{E}P_o \tag{9}$$

This is obtained by considering the outer boundary approaches to infinity ($b \to \infty$). Then the amount of induced radial and tangential stresses can be obtained as:

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[C_1 - (1-2\nu)\frac{C_2}{r^2}\right] \Rightarrow \sigma_r = P_o - (P_o - P_i)\left(\frac{a}{r}\right)^2 \tag{10}$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[C_1 + (1-2\nu)\frac{C_2}{r^2}\right] \Rightarrow \sigma_\theta = P_o + (P_o - P_i)\left(\frac{a}{r}\right)^2 \tag{11}$$

The induced stresses around a circular opening as a result of excavation and changing the initial stress condition (decreasing the internal boundary pressure from P_o to P_i) are solved. These equations have been used in many papers in rock engineering such as Brown [34]. In the internal boundary ($r = a$), the induced stresses are $\sigma_r = P_i$ & $\sigma_\theta = 2P_o - P_i$. In the case of zero internal pressure ($P_i = 0$), the stresses will be $\sigma_r = 0$ & $\sigma_\theta = 2P_o$, which is a very well-known stress concentration around circular tunnel in the hydrostatic stress condition.

Now the aim is to calculate directly the net displacement in the plane strain condition. As it was mentioned, the net displacement as a result of excavation occurs due to loading (in tangential direction) and unloading (in radial direction) in comparison to the initial stress conditions. This means that in the stress-strain relations, $\Delta\sigma_r = \sigma_r - P_o$ and $\Delta\sigma_\theta = \sigma_\theta - P_o$ (the stress difference factors that cause the net displacement) should be used. Considering the plane strain condition ($\varepsilon_z = 0, \sigma_z = \nu(\sigma_r + \sigma_\theta)$) and Equations (2) and (3), it can be written that:

$$\frac{du}{dr} = \frac{1}{E}[(1-\nu^2)\Delta\sigma_r - \nu(1+\nu)\Delta\sigma_\theta]$$
$$\frac{u}{r} = \frac{1}{E}[(1-\nu^2)\Delta\sigma_\theta - \nu(1+\nu)\Delta\sigma_r] \tag{12}$$

Using Equations (10) and (11) and $\Delta\sigma_r$ and $\Delta\sigma_\theta$, the second relation of Equation (12) can be written as:

$$\frac{u}{r} = \frac{1+\nu}{E}\left[(1-\nu)(P_o - P_i)\left(\frac{a}{r}\right)^2 + \nu(P_o - P_i)\left(\frac{a}{r}\right)^2\right] \Rightarrow u = \frac{1+\nu}{E}(P_o - P_i)\frac{a^2}{r} \tag{13}$$

Therefore, the net displacement can be obtained using Equation (13), which has been used in papers such as Brown [34]. In the tunnel boundary, it can be written that $r = a \Rightarrow u = \frac{1+\nu}{E}(P_o - P_i)a$. Equation (13) differs with the thick wall hallow cylinder (Lame' solution) displacement, as presented in the books by Popov [10] and Ameen [11], in which the total stresses has been considered (displacement as a result of internal and external boundary loading only, without considering the initial stresses effects). The procedure developed by Airy [35] and described by Timoshenko and Goodier [15] in establishing a particular form of the field equation for isotropic elasticity and plane strain can be followed to solve the problem. The Airy stress function formulation is based on the general idea of developing a representation for the stress field that satisfies equilibrium and yields a single governing equation from the compatibility statement. In the polar coordinate, the stress components in terms of an Airy stress function $\phi(r,\theta)$ were defined as [2, 13]:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{14}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}$$

$$\tau_{xy} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

From these equations and the compatibility equations in term of stresses, a fourth-order biharmonic partial differential equation in term of the function ϕ in cylindrical coordinate can be derived [15]:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \quad or \quad \nabla^4 \phi = 0 \qquad (15)$$

The solution to this equation can be obtained by the method of separation of variables such as $\phi(r,\theta) = R(r).\psi(\theta)$, where $R(r)$ is a function of r only, and $\psi(\theta)$ is a function of θ only. The details of the complete solution have been given by Little [36], although the original development is credited to Michell [37]. By selecting a suitable stress function that satisfies this equation and considering the appropriate boundary condition, the biharmonic equation can be solved. Then the stress components (stress fields around an opening) will be defined [13, 15]. For the axi-symmetric problem shown in Figure 1, the stress distributions do not depend on θ, which means that the derivatives of ψ with respect to θ is zero. Thus the following equation will be resulted from Equation 15:

$$\nabla^4 \phi = 0 \;\Rightarrow\; \frac{d^4 \phi}{dr^4} + \frac{2}{r}\frac{d^3 \phi}{dr^3} - \frac{1}{r^2}\frac{d^2 \phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0 \qquad (16)$$

This is an ordinary fourth-order differential equation. Using the same solution of Equation (5), the general solution can be obtained in the form of:

$$\phi = A\ln r + Br^2 \ln r + Cr^2 + D \qquad (17)$$

In this expression, the constants A, B, C, and D are determined by considering both the requirement for uniqueness of displacements and the pressure boundary conditions for the problem. The stress components are given by [7]:

$$\sigma_r = \frac{A}{r^2} + B(2\ln r + 1) + 2C$$
$$\sigma_\theta = -\frac{A}{r^2} + B(2\ln r + 3) + 2C \qquad (18)$$
$$\tau_{r\theta} = 0$$

It can be shown that uniqueness of displacements requires $B = 0$, and the stress components are $\sigma_r = \frac{A}{r^2} + 2C$ and $\sigma_\theta = -\frac{A}{r^2} + 2C$. Considering the boundary condition as $r = a \Rightarrow \sigma_r = P_i$ and $r = b \Rightarrow \sigma_r = P_o$, the constants can be obtained as:

$$A = \frac{P_o - P_i}{a^2 - b^2} a^2 b^2$$
$$2C = \frac{P_i a^2 - P_o b^2}{a^2 - b^2} \qquad (19)$$

Therefore, the induced radial and tangential stresses can be determined as:

$$\sigma_r = \frac{1}{r^2}\frac{P_o - P_i}{a^2 - b^2} a^2 b^2 + \frac{P_i a^2 - P_o b^2}{a^2 - b^2} \qquad (20)$$

$$\sigma_\theta = -\frac{1}{r^2}\frac{P_o - P_i}{a^2 - b^2} a^2 b^2 + \frac{P_i a^2 - P_o b^2}{a^2 - b^2} \qquad (21)$$

These are the stress distribution in a thick wall hollow cylinder (Lame' problem). Since $\tau_{r\theta}$ is zero, the stress components σ_r and σ_θ are the principal stresses. As it is considered that the tunnel is located in an infinite medium, the limits if these equation when $b \to \infty$ can be calculated as follow:

$$\sigma_\theta = P_o + (P_o - P_i)\left(\frac{a}{r}\right)^2$$
$$\sigma_r = P_o - (P_o - P_i)\left(\frac{a}{r}\right)^2 \qquad (22)$$

These are the same as Equations (10) and (11), obtained from the Lame's solution method. As explained in the previous section, the net displacement can be obtained using Equations (12) and (13). The general procedure to solve this problem (Figure 1) has been presented in references such as Obert and Duvall [2], Ameen [11], and Sadd [13]; however, none of them have calculated the net displacement.

Developed by Kolosov [38] and Muskhelishvili [39], the complex variable method provides a very powerful tool for the solution of many problems in elasticity. In this method, the displacements and stresses are represented in terms of two analytical functions of a complex variable. It is shown that the Airy stress function can be expressed as the real part of two analytic functions of a complex variable z [12, 13]. The procedure of obtaining the Airy stress function in the complex form is presented in Appendix 1, following the procedure presented by Sadd [13] and in a different way that Jaeger and Cook [12] presented. The Airy stress function can be presented in the following complex form:

$$\phi = \phi(z, \bar{z}) = \frac{1}{2}\left(z\overline{\gamma(z)} + \bar{z}\gamma(z) + \chi(z) + \overline{\chi(z)}\right) = \mathrm{Re}(\bar{z}\gamma(z) + \chi(z)) \qquad (23)$$

where $\gamma(z)$ and $\chi(z)$ are the arbitrary complex functions, the bar denotes the conjugate complex, and Re is the real part of the different terms.
As it is presented in Appendix 1, the stresses and displacements can also be presented based on the assumed complex functions as:

$$\sigma_x + \sigma_y = 2\left(\gamma'(z) + \overline{\gamma'(z)}\right) = 4\mathrm{Re}(\gamma'(z)), \qquad (24)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2(\bar{z}\gamma''(z) + \psi'(z))$$

$$2G(u + iv) = \kappa\gamma(z) + z\overline{\gamma'(z)} - \overline{\psi(z)} \qquad (25)$$

Where $\psi(z) = \chi'(z)$, G is the shear modulus of medium, u and v are the displacements in the x and y directions, respectively, and $\kappa = 3 - 4\nu$ is for the plane strain condition.
Using the transformation laws, the stresses and displacements in the polar coordinates can be written as:

$$\sigma_r + \sigma_\theta = \sigma_x + \sigma_y \qquad (26)$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = \left(\sigma_y - \sigma_x + 2i\tau_{xy}\right)e^{2i\theta}$$

$$u_r + iv_\theta = (u + iv)e^{-i\theta} \qquad (27)$$

The solution to particular problems in a two dimension involves selection of the suitable forms of the analytic functions $\gamma(z)$ and$\psi(z)$. Many useful solutions involve the polynomials in z or z^{-1} [7]. To solve the problem depicted in Figure 1, the following arbitrary analytical functions can be chosen:

$$\gamma(z) = cz \qquad (28)$$

$$\psi(z) = {}^{d}/_{z}$$

where c and d are the constants that can be in a complex form, and will be defined based on the boundary conditions. Based on Equations (24) and (26), it can be written that:

$$\sigma_r + \sigma_\theta = 2\left(\gamma'(z) + \overline{\gamma'(z)}\right) = 4\mathrm{Re}(\gamma'(z)) = 4\mathrm{Re}(c) \qquad (29)$$

$$\sigma_\theta - \sigma_r + 2i\tau_{r\theta} = \left(\sigma_y - \sigma_x + 2i\tau_{xy}\right)e^{2i\theta} = -\frac{2d}{z^2}e^{2i\theta} = -\frac{2d}{r^2} \qquad (30)$$

$$(z = re^{i\theta})$$

Equating the real and imaginary parts will result in $\tau_{r\theta} = 0$ and $\sigma_\theta - \sigma_r = -\frac{2d}{r^2}$. By considering the boundary conditions as $r = a \Rightarrow \sigma_r = P_i$ & $r = b = \infty \Rightarrow \sigma_r = P_o$, the following equation (σ_r) should be solved to find the constants:

$$\sigma_\theta - \sigma_r = -\frac{2d}{r^2} \qquad (31)$$

$$\sigma_\theta + \sigma_r = 4c \Rightarrow \sigma_r = 2c + \frac{d}{r^2}$$

This will give the constants as:

$$2c = \frac{b^2P_o - a^2P_i}{b^2 - a^2} \qquad (32)$$

$$d = \frac{(P_i - P_o)a^2b^2}{b^2 - a^2}$$

This is the same as Equation (19). Then the radial and tangential induced stresses can be given as:

$$\sigma_r = \frac{b^2P_o - a^2P_i}{b^2 - a^2} + \frac{(P_i - P_o)a^2b^2}{b^2 - a^2} \times \frac{1}{r^2} \qquad (33)$$

$$\sigma_\theta = \frac{b^2P_o - a^2P_i}{b^2 - a^2} - \frac{(P_i - P_o)a^2b^2}{b^2 - a^2} \times \frac{1}{r^2}$$

This is the same as Equations (20) and (21). By considering the tunnel excavation in an infinite medium ($b \to \infty$), the following equations will be resulted, which are the same as Equations (10) and (11).

$$\sigma_r = P_o - (P_o - P_i)\left(\frac{a}{r}\right)^2 \qquad (34)$$

$$\sigma_\theta = P_o + (P_o - P_i)\left(\frac{a}{r}\right)^2$$

Equations (25) and (27) can be used to evaluate the displacement, as Jaeger and Cook [12] have explained in the general solution procedure. However, this will not give the net displacement. As explained in the previous section, the net displacement can be obtained using the explained procedure using Equations (12) and (13).

4. Solutions for non-hydrostatic in situ stress

In the previous sections, the hydrostatic in situ stress condition has been considered. In this section, the problem shown in Figure 2 will be considered with zero internal pressure. The solution of this problem will yield the Kirsch equations. To solve this problem, the Airy stress function and the complex variable methods will be used (the Lame method cannot be used to solve this problem). Then the procedure for evaluating the net displacements will be presented.
The general solution of this problem using the Airy stress function method has been presented by Obert and Duvall [2] using the complex variable method given by Jaeger and Cook [12], and using both methods explained by Sadd [13]. However, the net displacements have not been given the same as the

Kirsch solution. In this section, the solutions will be reviewed, and this key point will be discussed. Finally, the internal pressure effect will also be considered.

4.1. Airy stress function method

At first, the problem depicted in Figure 2 with zero internal pressure is considered. Then the effect of internal pressure in a circular hole in an infinite medium is superimposed. At a large distance from the opening, the polar components of stress will be those resulting from the initial stress only. Considering the stress transformation laws, the stresses at $r=\infty$ are (boundary conditions at infinity):

$$\begin{cases} (\sigma_r)_{r=\infty} = \dfrac{P_h+P_v}{2} + \dfrac{P_h-P_v}{2}\cos 2\theta \\ (\sigma_\theta)_{r=\infty} = \dfrac{P_h+P_v}{2} - \dfrac{P_h-P_v}{2}\cos 2\theta \\ (\tau_{r\theta})_{r=\infty} = -\dfrac{P_h-P_v}{2}\sin 2\theta \end{cases} \quad (35)$$

The boundary conditions at tunnel boundary ($r=a$) are:

$$(\sigma_r)_{r=a} = (\tau_{r\theta})_{r=a} = 0 \quad (36)$$

Since the stress distributions depend on θ, the following Airy stress function may be selected:

$$\phi = A'\log r + B'r^2 + (C'r^2 + D'r^4 + E'r^{-2} + F')\cos 2\theta \quad (37)$$

where A', B', C', D', E', F' are constants, which will be determined from the boundary conditions. The stress components can be determined as:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{A'}{r^2} + 2B' + (-2C' - 6E'r^{-4} - 4Fr^{-2})\cos 2\theta \quad (38)$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} = -\frac{A'}{r^2} + 2B' + (2C' + 12D'r^2 + 6E'r^{-4})\cos 2\theta \quad (39)$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = (2C' + 6D'r^2 - 6E'r^{-4} - 2Fr^{-2})\sin 2\theta \quad (40)$$

Combining these equations with the boundary conditions, the constants will be determined, and then the following equations yield:

$$\sigma_r = \frac{P_h+P_v}{2}\left(1-\frac{a^2}{r^2}\right) + \frac{P_h-P_v}{2}\left(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta \quad (41)$$

$$\sigma_\theta = \frac{P_h+P_v}{2}\left(1+\frac{a^2}{r^2}\right) - \frac{P_h-P_v}{2}\left(1+\frac{3a^4}{r^4}\right)\cos 2\theta \quad (42)$$

$$\tau_{r\theta} = -\frac{P_h-P_v}{2}\left(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta \quad (43)$$

These are the well-known Kirsch equations for the induced stresses.

As it was explained, the net displacements occur due to loading and unloading in comparison to the initial stress conditions. This means that in the stress-strain relations, the stress differences ($\Delta\sigma_r = \sigma_r - \sigma_r^\infty$, $\Delta\sigma_\theta = \sigma_\theta - \sigma_\theta^\infty$ and $\Delta\tau_{r\theta} = \tau_{r\theta} - \tau_{r\theta}^\infty$) should be used, and thus:

$$\Delta\sigma_r = -\frac{P_h+P_v}{2}\frac{a^2}{r^2} + \frac{P_h-P_v}{2}\left(\frac{3a^4}{r^4}-\frac{4a^2}{r^2}\right)\cos 2\theta \quad (44)$$

$$\Delta\sigma_\theta = \frac{P_h+P_v}{2}\frac{a^2}{r^2} - \frac{P_h-P_v}{2}\left(\frac{3a^4}{r^4}\right)\cos 2\theta \quad (45)$$

$$\Delta\tau_{r\theta} = -\frac{P_h-P_v}{2}\left(-\frac{3a^4}{r^4}+\frac{2a^2}{r^2}\right)\sin 2\theta \quad (46)$$

The strain-displacement relations in a polar system are:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} \quad (47)$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}$$

Considering the plane strain condition and Equation (47), it can be written that:

$$\frac{\partial u_r}{\partial r} = \frac{1+\nu}{E}[(1-\nu)\Delta\sigma_r - \nu\Delta\sigma_\theta] \quad (48)$$

$$\frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = \frac{1+\nu}{E}[(1-\nu)\Delta\sigma_\theta - \nu\Delta\sigma_r] \quad (49)$$

$$\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} = \frac{2(1-\nu)}{E}\Delta\tau_{r\theta} \quad (50)$$

Integration of these equations and finding the integration constants based on the boundary conditions (the constants will be zero following the same procedure presented by Obert and Duvall), the displacements are given by:

$$u_r = \frac{1+\nu}{E}\frac{a^2}{r}\left[\frac{P_h+P_v}{2} + \frac{P_h-P_v}{2}\left(4(1-\nu)-\frac{a^2}{r^2}\right)\cos 2\theta\right] \quad (51)$$

$$v_\theta = -\frac{1+\nu}{E}\frac{P_h-P_v}{2}\frac{a^2}{r}\left[2(1-2\nu)+\frac{a^2}{r^2}\right]\sin 2\theta \quad (52)$$

where u_r is the radial inward displacement and v_θ is the tangential displacement (Figure 2). These are the net displacement relations (Kirsch solution) around a circular opening, which are different from the relations presented by Obert and Duvall [2] and Jaeger and Cook [12]. It should be noted that due to the assumed directions for the displacements, in some references such as Hudson [6], there is a negative sign difference in the u_r and v_θ equations.

In the case that there is an internal pressure, P_i, superposition can be used to evaluate the induced stresses and displacements. Simply, consider a pressurized hole in an infinite medium. The solution to this problem can be easily determined from the general case of the Lame' problem by choosing $P_o = 0$ and $b \to \infty$ (Figure 1). Thus the following equations can be written:

$$\sigma_r = P_i\left(\frac{a}{r}\right)^2$$

$$\sigma_\theta = -P_i\left(\frac{a}{r}\right)^2 \tag{53}$$

$$u = -\frac{1+\nu}{E}P_i\frac{a^2}{r}$$

These relations can be superimposed to Equations (41), (42), and (51), respectively, for considering the effect of the internal pressure in the Kirsch solution.

4.2. Complex variable method

The problem depicted in Figure 2 with zero internal pressure is considered. Following the analytical functions, it can be assumed that:

$$\gamma(z) = \frac{P_h+P_v}{4}\left(z+\frac{A''}{z}\right)$$

$$\psi(z) = \chi'(z) = \frac{P_v-P_h}{2}\left(z+\frac{B''}{z}+\frac{C''}{z^3}\right) \tag{54}$$

where A'',B'',C'' are the real constants. By differentiation with respect to $z(z = re^{i\theta})$, γ'_z, $\overline{\gamma'_z}$, γ''_z, and ψ'_z can be obtained. Based on Equations (24) and (26), and using the Euler formula ($e^{i\theta} = \cos\theta + i\sin\theta$), the following equations yield:

$$\sigma_r + \sigma_\theta = (P_h+P_v)(1-\frac{A''}{r^2}\cos2\theta) \tag{55}$$

$$\sigma_\theta - \sigma_r = -(P_v-P_h)\frac{B''}{r^2} + \left[(P_v-P_h)+(P_h+P_v)\frac{A''}{r^2}-(P_v-P_h)\frac{3C''}{r^4}\right]\cos2\theta \tag{56}$$

$$\tau_{r\theta} = \left[-\frac{P_h+P_v}{2}\frac{A''}{r^2}+\frac{P_v-P_h}{2}\left(1+\frac{3C''}{r^4}\right)\right]\sin2\theta \tag{57}$$

Solving equations (55) and (56) simultaneously results:

$$\sigma_r = \frac{P_h+P_v}{2}+\frac{P_v-P_h}{2}\frac{B''}{r^2} - \left[(P_h+P_v)\frac{A''}{r^2} + \frac{P_v-P_h}{2}\left(1-\frac{3C''}{r^4}\right)\right]\cos2\theta \tag{58}$$

$$\sigma_\theta = \frac{P_h+P_v}{2}-\frac{P_v-P_h}{2}\left[\frac{B''}{r^2}+\left(-1+\frac{3C''}{r^4}\right)\cos2\theta\right] \tag{59}$$

For zero radial pressure on the opening wall at $r = a$, the boundary conditions are $\sigma_r = 0$ and $\tau_{r\theta} = 0$, which allow determination of the constants as:

$$A'' = -\frac{P_v-P_h}{P_h+P_v}2a^2$$

$$B'' = -\frac{P_h+P_v}{P_v-P_h}a^2 \tag{60}$$

$$C'' = -a^4$$

Finally, the so-called Kirsch stress equations yield the bi-axially in situ stress condition, the same as Equations (41), (42), and (43).

It should be noted that this solution is a little bit different from the solution presented by Jaeger and Cook, in which a uniaxially stressed plane was taken into account, and then superposition was used for considering the biaxial stress condition.

Using Equations (25) and (27), Jaeger and Cook [12] evaluated displacements around the opening, which are not net displacements (the displacements as a result of initial stresses should be subtracted). To the contrary, after calculating the induced stresses, Equations (47) through (52) can be used for determining the net displacements. As explained earlier, the internal pressure effect can be considered using the superposition theorem.

5. Examples

In order to demonstrate the presented analytical method for determining the net displacement directly and show the differences with the analytical methods presented in the aforementioned references (for displacement around a tunnel), some examples are solved with assumed rock mass parameters and in situ stresses. As a case in example, a tunnel (a = 1 m) situated in the hydrostatic condition (Po = 10 MPa) with internal pressure (Pi = 0.5 MPa), and elastic rock mass parameters as E = 5000 MPa, and ν =

0.25 are considered. As shown in Figure 3, the analytical solutionss presented in the references give a total displacement that comprises the initial stress effects on the displacement (initial displacement). This displacement increases with increase in the distance from the tunnel wall, which can be misleading. To the contrary, based on the presented method for directly determining the net displacement as a result of excavation, the net displacement is maximum in the tunnel wall and decreases inside the rock mass. The initial displacements as a result of in situ stresses can be determined by subtracting the net displacement from the total one.

In order to compare the analytical results with the numerical results (for the purpose of validation of the proposed solution), a numerical modelling was done using the Phase 2 finite element software. The 4-Nodes quadrilateral mesh type with small dimensions are considered to get the accurate results [40]. In Figure 3, the results of the numerical analyses are shown along with the analytical results, which shows a good agreement.

It is worthwhile to mention that in the numerical modeling, an initial displacement also occurs, which is usually set to zero to get the net displacements as a result of excavation only.

To see the effect of rock mass elastic parameter change on the displacements, two different rock mass types are considered. The result is shown in Figure 4. As it can be seen, the general trends for the net displacement are similar but the amount is different. The same procedure can be shown for the non-hydrostatic in situ stress conditions.

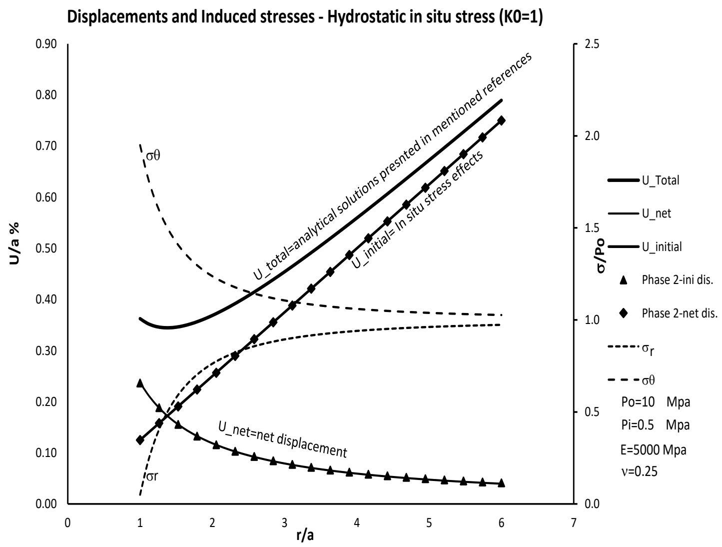

Figure 3. Net elastic displacement and total elastic displacements as a result of the excavation-hydrostatic in situ stress.

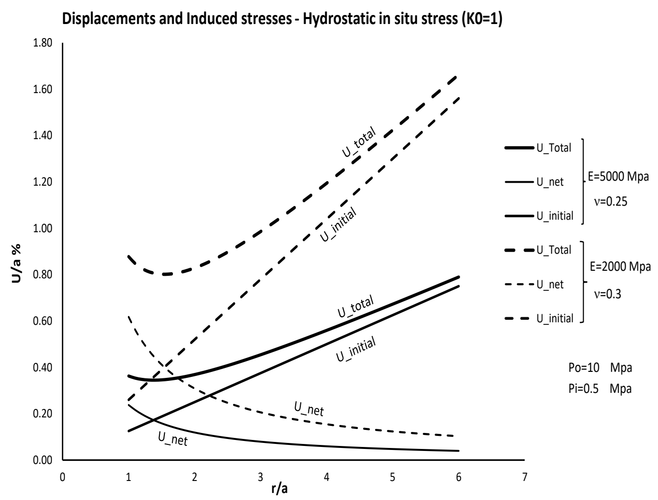

Figure 4. Displacements around a circular tunnel with two assumed rock mass types in the hydrostatic in situ stress.

6. Conclusions

Linear elasticity has been applied in many aspects of rock mechanics in order to understand the basic concepts, especially the stress and strain relationships such as their distribution around an excavated tunnel. However, there are some subtle points that can be misleading in its application in geomechanics. Directly determining the elastic net displacements, as a case in example, was presented in this work. Although the procedure for calculating the displacements and induced stresses around a circular tunnel in elastic condition is given in the literature, the net displacements have not been highlighted or even presented directly. Moreover, in some literature, as a result of considering the effect of the in situ (initial) stresses, the presented equations for calculating displacements around a circular tunnel differ from the Kirsch equations, which can be misleading in tunneling. This happens by mainly neglecting the geomechanics viewpoint in the application of linear elasticity.

In this paper, determining the induced stresses and net displacements around a circular tunnel in a biaxially loaded plate of homogeneous, isotropic, continuous, linearly elastic material in plain strain condition are reviewed and presented using different methods analytically and with the application of linear elasticity. Three solution methods including the Lame' method (hollow cylinder using strength of material), airy stress function method, and complex variable method are used for both the hydrostatic and non-hydrostatic in situ stresses in 2D biaxial loading condition. Presenting the Kirsch solution comprehensively, both loading and unloading in the tangential and radial directions, respectively, in tunneling (differences between induced stresses and initial stresses as a result of circular tunnel excavation) has been considered in equations, based on which, the net displacements can be calculated directly. Although most of the mathematical solutions may be found in the reference textbooks involving elasticity, these methods are presented by emphasis on loading and unloading around circular opening as a result of excavation. Actually, it is highlighted that in tunneling instead of only mechanical viewpoint, the ground real condition due to the existence of in situ stresses before excavation should be considered in determining the net displacements.

References

[1]. Kirsch, C. (1898). Die theorie der elastizitat und die bedurfnisse der festigkeitslehre. Zeitschrift des Vereines Dtsch. Ingenieure; 42:797–807.

[2]. Obert, L. and Duvall, W.I. (1967). Rock mechanics and the design of structures in rock.

[3]. Bray, J. (1978). Analytical and computational methods in engineering rock mechanics. Taylor & Francis.

[4]. Hoek, E. and Brown, E.T. (1980). Underground excavations in rock. CRC Press.

[5]. Goodman, R.E. (1989). Introduction to rock mechanics. Wiley New York.

[6]. Hudson, J.A. and Harrison, J.P. (2000). Engineering rock mechanics: an introduction to the principles. Elsevier.

[7]. Brady, B.H.G. and Brown, E.T. (2013). Rock mechanics: for underground mining. Springer Science & Business Media.

[8]. Savin, G.N. (1961). Stress concentration around holes.

[9]. Poulos, H.G. and Davis, E.H. (1974). Elastic solutions for soil and rock mechanics. John Wiley.

[10]. Popov, E.P. and Balan, T.A. (1990). Engineering mechanics of solids. Prentice Hall Englewood Cliffs, NJ.

[11]. Ameen, M. (2005). Computational elasticity. Harrow Alpha Sci. Int. Ltd.

[12]. Jaeger, J.C., Cook, N.G.W. and Zimmerman, R. (2009). Fundamentals of rock mechanics. John Wiley & Sons.

[13]. Sadd, M.H. (2009). Elasticity: theory, applications, and numerics. Academic Press.

[14]. Nam, S.W. and Bobet, A. (2007). Radial deformations induced by groundwater flow on deep circular tunnels. Rock Mech. Rock Eng. 40 (1):23.

[15]. Timoshenko, S.P. and Goodier, J.N. (1970). Theory of Elasticity, 3rd Edn McGraw-Hill. New York.

[16]. Bobet, A. (2009). Elastic solution for deep tunnels. Application to excavation damage zone and rockbolt support. Rock Mech. Rock Eng. 42 (2):147–74.

[17]. Oreste, P.P. and Peila, D. (1996). Radial passive rockbolting in tunnelling design with a new convergence-confinement model. Int. J. rock Mech. Min. Sci. Geomech. Abstr. 33 (5):443–54.

[18]. Carranza-Torres, C. and Fairhurst, C. (1999). The elasto-plastic response of underground excavations in rock masses that satisfy the Hoek–Brown failure criterion. Int. J. Rock Mech. Min. Sci. 36(6):777–809.

[19]. Carranza-Torres, C. and Fairhurst, C. (2000). Application of the convergence-confinement method of tunnel design to rock masses that satisfy the Hoek-Brown failure criterion. Tunn. Undergr. Sp. Technol. 15 (2):187–213.

[20]. Sharan, S.K. (2003). Elastic–brittle–plastic analysis of circular openings in Hoek–Brown media. Int. J. Rock Mech. Min. Sci. 40 (6):817–24.

[21]. Sharan, S.K. (2005). Exact and approximate solutions for displacements around circular openings in elastic–brittle–plastic Hoek–Brown rock. Int. J. Rock Mech. Min. Sci. 42(4):542–9.

[22]. Sharan, S.K. (2008). Analytical solutions for stresses and displacements around a circular opening in a generalized Hoek–Brown rock. Int. J. Rock Mech. Min. Sci. 1(45):78–85.

[23]. Bobet, A. (2006). A simple method for analysis of point anchored rockbolts in circular tunnels in elastic ground. Rock Mech. Rock Eng. 39 (4):315.

[24]. Oreste, P. (2008). Distinct analysis of fully grouted bolts around a circular tunnel considering the congruence of displacements between the bar and the rock. Int. J. Rock Mech. Min. Sci. 45 (7):1052–67.

[25]. Carranza-Torres, C. (2009). Analytical and numerical study of the mechanics of rockbolt reinforcement around tunnels in rock masses. Rock Mech. Rock Eng. 42 (2):175–228.

[26]. Bobet, A. (2011). Einstein HH. Tunnel reinforcement with rockbolts. Tunn. Undergr. Sp. Technol. 26 (1):100–23.

[27]. Tan, C.H. (2016). Difference solution of passive bolts reinforcement around a circular opening in elastoplastic rock mass. Int. J. Rock Mech. Min. Sci. 81:28–38.

[28]. Tan, C.H. (2016). Passive bolts reinforcement around a circular opening in strain-softening elastoplastic rock mass. Int. J. Rock Mech. Min. Sci. (88):221–34.

[29]. Lazemi, H.A., Marji, M.F., Bafghi, A.R.Y. and Goshtasbi, K. (2013). Rock Failure Analysis of the Broken Zone Around a Circular Opening/Analiza pęknięcia skały w strefie naruszonej wokół otworu kolistego. Arch. Min. Sci. 58 (1):165–88.

[30]. Abdollahi, M.S., Najafi, M., Bafghi, A.Y. and Marji, M.F. (2019). A 3D numerical model to determine suitable reinforcement strategies for passing TBM through a fault zone, a case study: Safaroud water transmission tunnel, Iran. Tunn. Undergr. Sp. Technol. 88:186–99.

[31]. Einstein, H.H. and Schwartz, C.W. (1979). Simplified analysis for tunnel supports. J. Geotech. Geoenvironmental Eng. 105 (ASCE 14541).

[32]. Kreyszig, E. (2011). Advanced engineering mathematics. United States of America: Jhon Wiley & Sons. INC.

[33]. Simmons, G.F. (2016). Differential equations with applications and historical notes. CRC Press.

[34]. Brown, E.T., Bray, J.W., Ladanyi, B. and Hoek, E. (1983). Ground response curves for rock tunnels. J. Geotech. Eng. 109(1):15–39.

[35]. Airy, G.B.I.V. (1863). On the strains in the Interior of beams. Philos. Trans. R. Soc. London. 153:49–79.

[36]. Little, R.W. (1973). Elasticity Prentice-Hall. Englewood Cliffs, NJ. 88–94.

[37]. Michell, J.H. (1899). On the direct determination of stress in an elastic solid, with application to the theory of plates. Proc. London Math. Soc. 1 (1):100–24.

[38]. Kolosov, G.V. (1909). On the Application of the Theory of Functions of a Complex Variable to a Plane problem in the Mathematical Theory of Elasticity. Printing-house K Mattisena, Juriev.

[39]. Muskhelishvili, N.I. (1966). Some basic problems of the mathematical theory of elasticity. Nauka, Moscow.

[40]. Rocscience. Phase 2, www.rocscience.com. (2004).

Appendix

The Sadd (2009) procedure is used to present the Airy stress function and stress components in complex forms. In complex functions, the following relations exist:

$$z = x + iy,\ \bar{z} = x - iy \Longrightarrow x = \frac{1}{2}(z + \bar{z}),\ y = \frac{1}{2}(z - \bar{z})$$

To differentiate a complex function relative to x and y:

$$\frac{\partial()}{\partial x} = \frac{\partial()}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial()}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x} = \frac{\partial()}{\partial z} + \frac{\partial()}{\partial \bar{z}}, \quad \frac{\partial()}{\partial y} = \frac{\partial()}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial()}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial y} = i(\frac{\partial()}{\partial z} - \frac{\partial()}{\partial \bar{z}})$$

The Laplacian operator in complex form is:

$$\nabla^2() = \frac{\partial^2()}{\partial x^2} + \frac{\partial^2()}{\partial y^2} = \frac{\partial}{\partial x}(\frac{\partial()}{\partial x}) + \frac{\partial}{\partial y}(\frac{\partial()}{\partial y}) \Longrightarrow$$

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial()}{\partial z} + \frac{\partial()}{\partial \bar{z}}\right) - \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\left(\frac{\partial()}{\partial z} - \frac{\partial()}{\partial \bar{z}}\right) = 4\frac{\partial^2()}{\partial z \partial \bar{z}}$$

This is the harmonic operator. Then the biharmonic operator will be $\nabla^2\nabla^2() = 16\frac{\partial^2()}{\partial z^2 \partial \bar{z}^2}$.

The idea is to represent the Airy stress function in terms of functions of a complex variable and transform the plane problem into one involving the complex variable theory. Using the above relations, the variables x and y can be expressed in terms of z and $\bar{z}$, and thus functions of x and y can be expressed as functions of z and $\bar{z}$. Applying this concept to the Airy stress function, it can be written that $\phi = \phi(z, \bar{z})$. Therefore, the governing biharmonic elasticity equation is $\frac{\partial^4 \phi}{\partial z^2 \partial \bar{z}^2} = 0$.

To solve this equation, integration should be done. In each stage, the integration constant will be considered as a function of the other variable, as follow:

$$\frac{\partial^3\phi}{\partial z^2\partial\bar{z}} = a(z) \Rightarrow \frac{\partial^2\phi}{\partial z\partial\bar{z}} = \int a(z)dz + b(\bar{z}) = c(z) + b(\bar{z}) \Rightarrow \frac{\partial\phi}{\partial z} = \bar{z}c(z) + \int b(\bar{z})d\bar{z} + d(z) = \bar{z}c(z) + e(\bar{z}) + d(z) \Rightarrow \phi = \bar{z}\int c(z)dz + ze(\bar{z}) + \int d(z)dz + f(\bar{z}) = \bar{z}g(z) + ze(\bar{z}) + h(z) + f(\bar{z})$$

Since the Airy stress function (ϕ) should be real, thus:

$$\phi = \mathrm{Re}[\bar{z}g(z) + ze(\bar{z}) + h(z) + f(\bar{z})] \Rightarrow$$
$$\mathrm{Re}[\bar{z}g(z)] = \frac{1}{2}[\bar{z}g(z) + zg(\bar{z})],\ \mathrm{Re}[ze(\bar{z})] = \frac{1}{2}[ze(\bar{z}) + \bar{z}e(z)]$$
$$\mathrm{Re}[h(z) + f(\bar{z})] = \frac{1}{2}[h(z) + h(\bar{z}) + f(\bar{z}) + f(z)] \Rightarrow$$
$$\phi = \frac{1}{2}[\bar{z}g(z) + zg(\bar{z}) + ze(\bar{z}) + \bar{z}e(z) + h(z) + h(\bar{z}) + f(\bar{z}) + f(z)] \Rightarrow$$
$$\phi = \frac{1}{2}(z(g(\bar{z}) + e(\bar{z})) + \bar{z}(g(z) + e(z)) + (h(z) + f(z)) + (h(\bar{z}) + f(\bar{z})))$$

Considering $\gamma(z) = g(z) + e(z)$ and $\chi(z) = h(z) + f(z)$, the Airy stress function in complex form will be $\phi(z,\bar{z}) = \frac{1}{2}\left(z\overline{\gamma(z)} + \bar{z}\gamma(z) + \chi(z) + \overline{\chi(z)}\right) = \mathrm{Re}(\bar{z}\gamma(z) + \chi(z))$.

This is Equation (27) presented in the paper. To obtain the stress components:

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = -\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right)\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right)\phi = -\left(\frac{\partial^2\phi}{\partial z^2} - 2\frac{\partial^2\phi}{\partial z\partial\bar{z}} + \frac{\partial^2\phi}{\partial\bar{z}^2}\right)$$
$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right)\phi = \left(\frac{\partial^2\phi}{\partial z^2} + 2\frac{\partial^2\phi}{\partial z\partial\bar{z}} + \frac{\partial^2\phi}{\partial\bar{z}^2}\right)$$
$$\sigma_x + \sigma_y = 4\frac{\partial^2\phi}{\partial z\partial\bar{z}} = 2\left[\gamma'(z) + \overline{\gamma'(z)}\right]$$
$$\sigma_y - \sigma_x = 2\left(\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial\bar{z}^2}\right)$$
$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -i\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\bar{z}}\right)\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial\bar{z}}\right)\phi = -i\left(\frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial\bar{z}^2}\right)$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 4\frac{\partial^2\phi}{\partial z^2} = 2[\bar{z}\gamma''(z) + \psi'(z)]$$

روش‌های تحلیلی تعیین جابجایی‌های الاستیک خالص ناشی از حفاری در اطراف تونل دایره‌ای

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چکیده:

جابجایی‌های اطراف تونل که در اثر حفاری رخ می‌دهند شامل جابجایی الاستیک و پلاستیک می‌باشند. در این مقاله جابجایی‌های الاستیک ناشی از حفاری که جابجایی خالص نیز خوانده می‌شوند مورد بحث قرار گرفته است. به طور کلی روش‌های تحلیلی ارائه شده قبلی برای تعیین جابجایی الاستیک اطراف تونل دایره‌ای، جابجایی خالص ناشی از حفاری را به طور مستقیم ارائه نمی‌کنند. روش معروف کرش پرکاربردترین روش برای تعیین تنشهای القایی و جابجایی خالص ناشی از حفاری اطراف تونل دایره‌ای است که در شرایط تنش دوبعدی، محیط الاستیک خطی و هموژن مورد استفاده قرار می‌گیرد. ولی چگونگی تعیین مقادیر جابجایی خالص به این روش در مراجع به صورت جامع ارائه نشده است. این مقاله با استفاده از تئوری الاستیسیته سه روش تحلیلی برای تعیین تنش‌های القایی و جابجایی خالص اطراف تونل دایره‌ای را مورد بررسی قرار داده است. این سه روش عبارتند از: روش لامه، روش تابع تنش ایری و روش تابع با متغیرهای مختلط. در این مقاله فرض شده است که تونل به صورت دایره‌ای در محیط پیوسته، الاستیک، هموژن و ایزوتروپ قرار دارد. این روشها برای دو حالت تنش اولیه برجای هیدرواستاتیک و غیر هیدرواستاتیک با یک فشار داخلی و شرایط بارگذاری دوبعدی ارائه شده است. بارگذاری و باربرداری ناشی از حفاری رخ داده در اطراف تونل به همراه اختلاف تنش القایی با تنش اولیه در تعیین جابجایی خالص مورد استفاده قرار گرفته است. در انتها به منظور نشان دادن اختلاف جابجایی خالص ناشی از حفاری و جابجایی کل که تاثیر تنش‌های اولیه را در نظر می‌گیرد، مثال‌های عددی حل شده و با نتایج روش تحلیلی مقایسه شده است.

کلمات کلیدی: الاستیسیته خطی، تونل دایره‌ای، جابجایی خالص، محیط الاستیک، روشهای تحلیلی.